Atiyah-Singer Index Theorem

and

Quantum Field Theory

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0. Introduction

Recently, the relevance of global features in quantum field theory has been much emphasized. These properties, which in perturbation theory at most reflect themselves in germinal form , are accessible by global differential geometric and topological methods and are of crucial importance even for the qualitative behaviour of the quantum system.

Much work has been done on the topological and, for gauge fields, also on the differential geometric classification [1] of the stationary configurations of the classical action, which are supposed to contain interesting information about the affiliated quantum field theory.

Here we shall concentrate on a different question, namely on the spectrum of fluctuations about a stationary point of a Euclidean action. This problem arises immediately, already to lowest order in Planck's constant $\hbar$, if one tries to quantize a classical field theory. In mathematical terms it corresponds to the determination of the eigenvalues of an elliptic differential operator. It is known that this eigenvalue distribution reflects fundamental topological and also metrical features of the underlying system. For instance the area, boundary length and number of holes of a swinging membrane can be obtained from the spectrum of its oscillations, it is, indeed, largely possible to "hear the shape of a drum" [2]. The metrical and topological properties of the spectrum of quantum fluctuations are of immediate physical relevance. A beautiful example is provided by the Casimir effect [3], an attraction between two uncharged conducting plates, which is due to the topological modification of the vacuum fluctuations as compared to the vacuum without plates. Closely related [4][5] to this is the trace anomaly, a quantum effect for the energy momentum tensor which depends on both topological and metric invariants, namely the Euler characteristic and certain integral curvatures.

The Atiyah-Singer theorem [6] states that a very special spectral feature, the difference between the number of zero modes of an elliptic operator and its adjoint, the so-called index of the operator has a purely topological meaning and is related to the (differential) topology of the underlying base (space-time) manifold and the winding numbers of the bundles which appear in the problem.

On the other hand, the index can be shown [7] to be related to the anomaly of an associated current, so that current anomalies turn out to have topological significance.

The connecting link between the metrical and topological properties of a system and the spectrum of an elliptic operator D is the asymptotic expansion of the heat transport kernel of the operators D^*D and $D\,D^*$ [8], which, on the physical side is closely related to certain renormalization schemes like analytic renormalization point splitting [9][5] [19], ζ-function renormalization [4][8][10] and Schwinger's proper time formalism[5].

In this work we shall mainly deal with physical applications of the Atiyah-Singer index theorem. The difficulty in the presentation lies in the fact that most of the mathematical notions employed like manifolds [11], bundles [11] [12], characteristic classes [13] [6], cohomology etc., although intuitive and directly interpretable in physical terms are not so well-known to physicists, and that a reasonably complete description of these notions would lead to an unbalance of the mathematical and physical part of this work.
We try to cope with this difficulty by simply referring to the available good literature for the notions of manifolds [11] and fibre bundles [11][12] and by explaining the contents of the Atiyah-Singer theorem [6] in as non-technical a way as possible, trying to exhibit the essential ideas without striving for completeness and full rigor and not even attempting to describe a proof of the theorem.

The organisation of this work is as follows:
In chapter one we state and explain the contents of the Atiyah-

Singer theorem.
Chapter two contains a description of the heat transport formalism [8], which is vital for understanding the connection between spectral and metrical topological properties and links up with renormalization theory.
In chapter three, as an immediate application of the index theorem, the dimension of the space of non gauge equivalent fluctuations about a self dual instanton configuration is evaluated [14 - 18]. This leads to a determination of the number of parameters of instanton solutions of Yang-Mills' equations for gauge fields an arbitrary compact four dimensional orientable Euclidean space-time manifolds with arbitrary simple gauge group.
Chapter four contains a description of the relationship [7] between current anomalies and index theorem for a large class of anomalous currents, thus establishing the topological significance of current anomalies.
This general insight is applied in chapter five to evaluate the gravitational part of the axial anomaly as well as the Yang Mills part for fermion fields of spin 1/2 [19] and also spin 3/2 [10] (supergravity). In addition new anomalous currents are constructed [19], whose anomalies are related to the Euler characteristic and the signature of Euclidean space time.
Finally chapter six deals with noncompact space time manifolds. The generalized Atiyah-Singer theorem for this case [20] contains a peculiar non-local boundary term, which helps to resolve a puzzle [21][22] about apparently fractional winding numbers. The index in gravitational background fields is evaluated for spin 1/2 [22] and spin 3/2 [23]. The index theorem is a powerful tool for calculating the additional boundary contribution. The case of spin 3/2 turns out to be particularly interesting because only there the index is nonvanishing, making chirality losses by vacuum tunnelling possible.

For the reader's convenience some essentials about de Rham cohomology, characteristic classes and formal splitting methods are collected in two short appendices.

1. The Atiyah-Singer Index Theorem

In this section we shall briefly explain the content and the meaning of the fundamental index theorem of Atiyah and Singer [6], a theorem on elliptic operators between complex vector bundles over a compact manifold. For all notions of manifolds and vector bundles we refer to the abundant literature [11)12)] on these subjects.

Let E and F be complex vector bundles over a manifold M. The sets of sections of E and F will be denoted by $\Gamma(E)$ and $\Gamma(F)$ respectively. For physical situations M can be interpreted e.g. as a Euclidean space-time manifold and the sections in $\Gamma(E)$ and $\Gamma(F)$ as suitable vector fields (spinors, isospinors etc...)

A differential operator D from E to F (written $E \xrightarrow{D} F$) is by definition a linear mapping

$$D: \quad \Gamma(E) \longrightarrow \Gamma(F) \; ,$$

which in terms of local coordinates in M and the fibres of E and F assumes the form

$$(D v(x))_i = \sum_{\substack{|\alpha| \leq m \\ j}} a^{\alpha}_{ij}(x) \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} v_j(x) \tag{1.1}$$

with $\quad 1 \leq i \leq \dim F, \quad 1 \leq j \leq \dim E$ (1.1a)

the fibre dimensions of E and F.
The summation on the right hand side of eq. (1.1) runs over the multiindex

$$\begin{aligned} \alpha &= (\alpha_1, \dots, \alpha_n); \quad n = \dim M \\ |\alpha| &= \alpha_1 + \alpha_2 + \dots + \alpha_n \end{aligned} \tag{1.1b}$$

The finite natural number m is called the <u>order</u> of the differential operator D. It does not depend on the coordinates chosen. To the operator D we associate a <u>symbol</u> σ_D by taking the "leading part" of the operator and substituting the derivatives $\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}$ by monomials $\xi^{\alpha} := \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n}$, $\xi_i \in \mathbb{R}$:

$$\sigma_D(x, \xi)_{ij} = \sum_{|\alpha| = m} a^{\alpha}_{ij}(x) \, \xi^{\alpha} \; . \tag{1.2}$$

The elements $\sigma_D(x, \xi)_{ij}$ define a dim F x dim E matrix. (It is easy to show that the symbol defines a vector bundle homomorphism $\sigma_D : \pi^* E \longrightarrow \pi^* F$, where π is the projection $\pi : T^* M \longrightarrow M$ of the cotangent bundle over M onto M).

The operator D is called <u>elliptic</u>, if the matrix $\sigma_D(x, \xi)$ is <u>invertible</u> for $\xi \neq 0$.

For instance, the Laplace operator on $\mathbb{R}^3$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, has a symbol $\sigma_\Delta(x,\xi) \;=\; \xi_1^2 + \xi_2^2 + \xi_3^2$ (one by one matrix) and is elliptic, whereas the d'Alembert operator on $\mathbb{R}^4$, $\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$, with symbol $\sigma_\Box = \xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2$ is not elliptic.

We now define the kernel of D as the space of "zero modes" of D:

$$\ker D = \{ v \in \Gamma(E) \mid Dv = 0 \} \tag{1.3}$$

the image of D

$$\mathrm{im}\, D = \{ w \in \Gamma(F) \mid \exists v \in \Gamma(E) \quad w = Dv \} \tag{1.4}$$

the set of sections in $\Gamma(F)$ of the form Dv with $v \in \Gamma(E)$ and the cokernel of D:

$$\mathrm{coker}\, D = \Gamma(F) / \mathrm{im}\, D \tag{1.5}$$

the quotient space of $\Gamma(F)$ modulo im D.
For compact M it is a fact of functional analysis [6) 24)] that ker D and coker D are finite dimensional.

Hence, the (analytic) index of the elliptic operator can be defined:

$$\mathrm{ind}\, D \;=\; \dim \ker D - \dim \mathrm{coker}\, D \quad . \tag{1.6}$$

Equivalently one can consider the adjoint D^* of D with respect to any hermitean metric in the fibres and define

$$\mathrm{ind}\, D \;=\; \dim \ker D - \dim \ker D^* \quad . \tag{1.6a}$$

The amazing statement of the Atiyah-Singer index theorem [6)] is now, that this apparently analytic quantity, defined by the number of solutions of certain linear partial differential equations actually turns out to have a topological meaning. It can also be obtained by evaluating a certain characteristic class [13)] which depends on the operator D and on the tangent bundle TM of M.

It is not possible at this place to give a full description of the concept of characteristic classes of (vector) bundles. Some essentials will be given in appendix II. Here we only mention that for a vector bundle E over a manifold M a characteristic class $\chi(E)$ is a cohomology class on M, which gives us information about the degree of nontriviality of the bundle E. For trivial E $\chi(E)$ is trivial, a nontrivial $\chi(E)$ implies nontriviality of E. The characteristic classes we shall need later on are furthermore representable [25] as closed differential forms which are polynomials in the curvature quantities of the bundle E, obtained by introducing some linear connection on E, in concrete cases polynomials in the Yang -Mills field strength and the Riemannian curvature tensor.

In the cases we are going to consider the Atiyah Singer theorem assumes the following form:

Let E and F be complex vector bundles over a compact n-dimensional manifold M (without boundary). Let $E \xrightarrow{D} F$ be an elliptic operator then

$$\mathrm{ind}\, D = (-1)^{\frac{n}{2}(n+1)} \frac{\mathrm{ch}\, E - \mathrm{ch}\, F}{e(TM)}\, td\,(TM \otimes \mathbb{C})\,[M] \quad . \qquad (1.7)$$

To arrive at this simple form we have assumed that D has, what is called a universal interpretation [6] which roughly corresponds to the physical statement that the operator can be defined by itself without reference to its concrete realization on the manifold M and the bundles E and F. The Dirac operator is an example of this kind of universality.

On the right hand side of eq. (1.7) ch E, ch F, e(TM), td($TM \otimes \mathbb{C}$) are the Chern characters of E and F, the Euler class of the tangent bundle TM and the Todd class of the complexified tangent bundle $TM \otimes \mathbb{C}$. They are all to be understood as closed differential forms of even degree which in turn are polynomials in the Yang Mills field strengths or the Riemannian curvature. The division by e(TM) can really be performed, and the result is a closed differential form on M, not homogeneous in its degree. The symbol [M] indicates that one has to extract the part of degree $n = \dim M$ and integrate it over the compact manifold M to obtain ind D.

Altogether ind D is obtained as the integral over a well defined polynomial in the curvature quantities of the bundles which appear in the problem. More precise definitions and details will be given in appendix II, where it will also be shown how the polynomial in the curvature quantities can readily be obtained from the theory of the characteristic classes. In the next section we shall also indicate a direct but laborious way of calculating the polynomial for a given operator.

Some remarks may be appropriate:

a) Trivially, for self-adjoint operators the index vanishes.

b) From (1.7) it is evident that in the cases considered the index does not depend on the operator but only on the bundles involved and, of course on the nontrivial fact that there is an elliptic operator connecting the two bundles E and F. (This requires for instance dim E = dim F.)

c) There is a useful apparent generalization of the A.S. theorem, which on close inspection turns out to be equivalent to the original formulation, but which is nevertheless worthwhile to formulate:

Take a finite sequence of complex bundles over a compact manifold and of differential operators

$$0 \xrightarrow{D_{-1}} E_0 \xrightarrow{D_0} E_1 \xrightarrow{D_1} E_2 \xrightarrow{D_2} \cdots\cdots \xrightarrow{D_{n-1}} E_n \xrightarrow{D_n} 0 \quad . \qquad (1.8)$$

This sequence is called an elliptic complex, if $D_i D_{i-1} = 0$ and if the sequence of symbols σ_{D_i} is exact, i.e.

$$\mathrm{im}\,\sigma_{D_{i-1}} = \ker \sigma_{D_i} \qquad \text{for all } i \quad . \qquad (1.9)$$

In this case the spaces

$$H_i = \ker D_i / \mathrm{im}\, D_{i-1} \qquad (1.10)$$

have finite dimensions, and, denoting the sequence (1.9) by D, one defines

$$\mathrm{ind}\, D = \sum_{i=0}^{n-1} (-1)^i \dim H_i \quad . \qquad (1.11)$$

Then, under the same universality condition as above, the Atiyah-Singer theorem says

$$\mathrm{ind}\, D = (-1)^{\frac{n}{2}(n+1)} \frac{\sum_{i=0}^{n} (-1)^i \mathrm{ch}\, E_i}{e(TM)} \; td(TM \otimes \mathbb{C})[M] \quad . \qquad (1.12)$$

Evidently the original theorem (1.7) is subsumed as the special case of a two step complex.

There are also generalizations of the index theorem to manifolds with boundary, to which we shall return in the last chapter.

We conclude this paragraph by mentioning another very important generalization of the index theorem which will be employed later on, the G index theorem [6).

Take again two complex vector bundles E and F over a compact manifold M. In addition we assume that a group G acts on M and on the bundles E and F in such a way that for every $m \in M$ the fibre of E over m is mapped linearly onto the fibre of E over gm, and analogously for F. Then G also acts on the sections of E (and F) in the following way:

$$s \longrightarrow gs \qquad (s, gs \in \Gamma(E))$$

with

$$(gs)(m) = g(s(g^{-1}m)) \quad . \tag{1.13}$$

We now consider an elliptic operator $E \xrightarrow{D} F$, which commutes with the action of G:

$$gD = Dg \qquad \text{for all } g \in G \tag{1.14}$$

Then ker D and coker D are stable under G and thus finite dimensional representation spaces of the group G. For every $g \in G$ one can form the traces of g on these representation spaces and define

$$\mathrm{ind}_g D = \mathrm{tr}\, g\big|_{\ker D} - \mathrm{tr}\, g\big|_{\mathrm{coker}\, D} \tag{1.15}$$

Then the G index theorem says that this quantity has topological meaning and is the value of a characteristic class. One has

$$\mathrm{ind}_g D = \frac{i^*(\mathrm{ch}_g E - \mathrm{ch}_g F)\cdot \mathrm{td}\,(TM^g)}{\mathrm{ch}_g(\Lambda_{-1} N^g \otimes \mathbb{C})\, e\,(TM^g)}\,[M^g] \tag{1.16}$$

where M^g is the set of fixed points of g, TM^g its tangent bundle, $N^g \otimes \mathbb{C}$ the complexified normal bundle of TM^g in TM, $\Lambda_{-1}(N^g \otimes \mathbb{C})$ the alternating sum of the external powers of $N^g \otimes \mathbb{C}$ and ch_g the so-called equivariant Chern character, a generalization of the usual Chern character, which will be defined in the appendix II.

We see that the G-index is character valued rather than integer valued and that the usual index theorem is recovered for $g = e$, the identity element of the group G.

2. Index Theorem and Heat transport [8)]

Consider a complex vector bundle E over a compact manifold M and a non-negative elliptic operator $\Delta : \Gamma(E) \longrightarrow \Gamma(E)$ on E. For this operator one constructs the heat transport operator

$$h = e^{-t\Delta} \tag{2.1}$$

The kernel of this operator has the spectral decomposition in terms of the eigenfunctions of Δ :

$$h(t,x,y) = \sum_{\lambda} e^{-\lambda t} \varphi_\lambda(x) \overline{\varphi_\lambda(y)} \tag{2.2}$$

where the summation runs over the eigenvalues of Δ , multiple eigenvalues being counted several times.

(For all of our constructions one has, strictly speaking to complete $\Gamma(E)$ to a Hilbert space with an appropriate Sobolev norm. Details can be found in ref. [24]).

The matrix valued kernel function obeys the heat transport equation

$$\left(\frac{\partial}{\partial t} + \Delta_x\right) h(t,x,y) = 0 \tag{2.3a}$$

with initial value

$$h(0,x,y) = \delta(x,y) \quad . \tag{2.3b}$$

For $t \longrightarrow \infty$ it simply tends to the projector onto the space of zero modes of Δ .

It can be shown [8) 24)] that for small positive t the function h(t,x,x) has an asymptotic expansion

$$h(t,x,x) \underset{t\to 0_+}{\sim} \sum_r t^r \mu_r(x) \tag{2.4}$$

with only finitely many negative powers of t. Furthermore the coefficient functions $\mu_r(x)$ can recursively be determined as polynomials in the coefficients of the operator Δ and their derivatives. For differential operators with a geometric meaning only curvature tensors and their derivatives occur. In concrete cases this recursive determination may be very laborious.

The related quantity $h(t) \overset{\text{Def}}{=} \operatorname{tr} \int_M dx\, h(t,x,x)$ equals

$$h(t) = \sum_{\lambda} e^{-\lambda t} \tag{2.5a}$$

and has an asymptotic expansion

$$h(t) \underset{t\to 0_+}{\sim} t^r a_r \tag{2.5b}$$

with

$$a_r = tr \int_M \mu_r(x)\,dx \quad . \tag{2.5c}$$

Sometimes it is advantageous to consider the operator Δ^{-s}, whose kernel $\zeta(s,x,y)$ is related to $h(t,x,y)$ by a Mellin transform

$$\zeta(s,x,y) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \left(h(t,x,y) - P_o(x,y) \right) \tag{2.6}$$

where $P_o(x,y) = \sum \psi_o^*(x) \cdot \psi_o(y) = \lim_{t \to \infty} h(t,x,y)$ is the projector onto the kernel of Δ.

And the related quantity

$$\zeta(s) = \sum_{\lambda \neq 0} \lambda^{-s} = tr \int_M dx\, \zeta(s,x,x) \tag{2.7}$$

$\zeta(s)$ is analytic for s large enough, and for $s = 0$ one has

$$\zeta(0) = a_o + \int_M dx\, P_o(x,x) \quad . \tag{2.8}$$

Now take an elliptic operator $D: \Gamma(E) \longrightarrow \Gamma(F)$ between two bundles E and F. With the adjoint D^* of D one observes that the operators

$$\begin{aligned} \Delta_E &= D^*D \; : \; \Gamma(E) \longrightarrow \Gamma(E) \\ \Delta_F &= DD^* \; : \; \Gamma(F) \longrightarrow \Gamma(F) \end{aligned} \tag{2.9}$$

are non-negative. Furthermore, they have the same eigenvalues λ, which, for $\lambda \neq 0$ even are of the same multiplicity. To see this, one only has to realize that for $\lambda \neq 0$ and

$$\begin{aligned} \Delta_E \varphi_\lambda &= D^*D \varphi_\lambda = \lambda \cdot \varphi_\lambda \\ \Delta_F (D\varphi_\lambda) &= (DD^*) D \varphi_\lambda = D(D^*D)\varphi_\lambda = \lambda D \varphi_\lambda \end{aligned} \tag{2.10}$$

and that $D\varphi_\lambda \neq 0$ because Δ_E and D have the same kernel. Constructing the corresponding heat kernels $h_E(t,x,y)$ and $h_F(t,x,y)$ one notices that the quantity

$$h_E(t) - h_F(t) = \sum_{\lambda_E} e^{-\lambda_E t} - \sum_{\lambda_F} e^{-\lambda_F t} \tag{2.11}$$

actually does not depend on t, because the contributions from the non-vanishing eigenvalues cancel out.
Hence, because $\ker D = \ker \Delta_E$ and $\ker D^* = \ker \Delta_F$

$$h_E(t) - h_F(T) = \dim \ker D - \dim \ker D^* = \operatorname{ind} D \quad . \qquad (2.12)$$

Inserting the asymptotic expansion (2.4) and noticing that only μ_o^E and μ_o^F can contribute , one finally obtains

$$\operatorname{ind} D = \operatorname{tr} \int_M (\mu_o^E(x) - \mu_o^F(x))\, dx = a_o^E - a_o^F \qquad (2.13)$$

a formula, which expresses the index of D as an integral over curvature quantities. The integrand is identical with the right-hand side of the Atiyah-Singer index theorem (1.7), if the characteristic classes are expressed in terms of curvature quantities. Of course, it is always advantageous to use (1.7) directly rather than going through all the recursion formulae which lead to (2.13), but the insight coming from the relationship of the index to the heat kernel will be useful for the application of the index theorem to current anomalies, which will be described in section 4.
We conclude with two remarks:
a) The coefficients μ_r of eq. (2.4) normally only have a differential geometric meaning. The Atiyah Singer theorem states that the special combination $\mu_o^E - \mu_o^F$ has topological meaning.
b) The relationship of the index problem to the heat transport equation is used for the heat transport proof of the index theorem. Here formula (2.13) is evaluated for a <u>certain set</u> of elliptic operators, and the resulting integrand identified as a characteristic class in terms of curvature quantities. Then K-theory is applied to show that this is sufficient to show that the index of <u>every</u> elliptic operator is the value of a characteristic class and to derive a formula of type eq. (1.7).

3. Number of Parameters of Instanton Solutions

We consider some Euclidean space time manifold M. We require M to be a four dimensional Riemannian manifold, compact and oriented. The common case in the literature [14-17] is $M = S^4$, the four dimensional sphere, but other base manifolds may be of similar interest, especially in quantum gravity.

On M we have a Yang -Mills field, which belongs to a compact

Lie group G and is given by a (locally defined) gauge potential $A^{*)}$, a one-form on M with values in the Lie algebra $\mathfrak{g}$ of G. In condensed notation the Yang -Mills field theory strength is

$$F = dA + \frac{1}{2} A \wedge A \tag{3.1}$$

a $\mathfrak{g}$-valued two-form on M. The wedge product $A \wedge A$ implies a Lie multiplication.

*) In mathematical terms we have a principal G-bundle P over M and a connection on P.

In our notation the Bianchi identity looks like

$$D_2 F \overset{\mathrm{Def}}{=} dF + A \wedge F = 0 \tag{3.2}$$

and the effect of an infinitesimal gauge transformation on A is

$$\delta A = d\chi + A \wedge \chi \overset{\mathrm{Def}}{=} D_0 \chi \tag{3.3}$$

where χ is some $\mathfrak{g}$-valued function on M (more precisely a section of the vector bundle $V_{\mathfrak{g}}$, the vector bundle with fibre $\mathfrak{g}$, associated to the principal bundle P via the adjoint representation of G.

An <u>instanton</u> over M is defined to be a connection A such that the corresponding Yang -Mills field strength is <u>self-dual</u> :

$$*F = F \quad . \tag{3.4}$$

This implies that the Yang-Mills equation

$$D_2 * F = (d + A \wedge) * F = 0 \tag{3.5}$$

is fulfilled.

The question is now what the dimensionality of a local family of instanton solutions is, if one identifies gauge equivalent instanton solutions.

The variation of F due to an infinitesimal variation δA is because of (3.1) given by

$$\delta F = d\delta A + A \wedge \delta A \overset{\mathrm{Def}}{=} D_1 \delta A \ . \tag{3.6}$$

So, an equivalent formulation of our question is: What is the dimension h_1 of the space of variations δA, which have F self-dual and are not of the form (3.3), i.e. not due to a mere gauge transformation?

Denoting the projector onto antiselfdual two-forms by P_- we can say, that we want to compute the quantity

$$h_1 = \ker P_- \circ D_1 / \mathrm{im}\, D_o \quad . \tag{3.7}$$

This quantity is well defined because, indeed $\mathrm{im}\, D_o \subset \ker P_- D_1$, because $D_1 D_o \chi = F\chi$ and hence

$$(P_- D_1) D_o \chi = 0 \tag{3.8}$$

because F was self dual. The index theorem now enters via the complex [16]

$$D: \quad 0 \longrightarrow \Lambda^o_{\mathfrak{g}}(M) \xrightarrow{D_o} \Lambda^1_{\mathfrak{g}}(M) \xrightarrow{P_- D_1} \Lambda^2_{-\mathfrak{g}}(M) \longrightarrow 0,$$

which can easily shown to be elliptic.

Here

$$\begin{aligned} \Lambda^o_{\mathfrak{g}}(M) &= \Lambda^o(M) \otimes V_{\mathfrak{g}} \\ \Lambda^1_{\mathfrak{g}}(M) &= \Lambda^1(M) \otimes V_{\mathfrak{g}} \\ \Lambda^2_{-\mathfrak{g}}(M) &= \Lambda^2_-(M) \otimes V_{\mathfrak{g}} \ , \end{aligned} \tag{3.9}$$

where $V_{\mathfrak{g}}$ was defined above and $\Lambda^o(M)$, $\Lambda^1(M)$, $\Lambda^2_-(M)$ are the zeroth, first and antiselfdual second exterior power of the cotangent bundle of M. The sections of these bundles are zero-forms, one-forms and antiselfdual two-forms respectively.

Now, according to (1.11)

$$\mathrm{ind}\, D = h_o - h_1 + h_2 \tag{3.10}$$

with h_1 given by (3.7) and

$$\begin{aligned} h_o &= \dim \ker D_o \\ h_2 &= \dim \ker (P_- D_1)^* = \dim(\Lambda^2_{-\mathfrak{g}}(M) / \mathrm{im}\, P_- D_1) \end{aligned} \tag{3.11}$$

ind D can be evaluated by the Atiyah-Singer index theorem, and one obtains

$$h_1 = 4kC(G) - 1/2 \dim G\, (\chi - \tau) + h_o + h_2 \quad , \tag{3.12}$$

Here k is the winding number of the Instanton

χ is the Euler characteristic of M

τ is the signature of M
C(G) and dim G depend on the group and are given by the following table [17]:

G	SU(n)	SO(n) n 7	Sp(2n)	G_2	F_4	E_6	E_7	E_8
C(G)	n	n-2	n+1	4	9	12	18	30

The quantities h_o and h_2 vanish in many interesting cases. h_o is non-vanishing only if the connection is actually reducible to a subgroup $H \subset G$.

A sufficient condition for $h_2 = 0$ is that the scalar curvature of M is non-negative and that the Weyl-tensor of M is selfdual. This also guarantees [10] that the local variations are integrable to global instanton solutions.

As an example one may take the well known case

$M = S^4$, hence $\chi = 2$, $\tau = 0$,

$G = SU(2)$ with $C(G) = 2$, $\dim G = 3$ (3.13)

and $h_o = h_2 = 0$.

Then the famous result $h_1 = 8k - 3$ (3.14)

is reproduced.

4. Index theorem and Current Anomalies

Current anomalies [26] are a characteristic phenomenon, which may arise if a classical field theory is quantized. The invariance of a classical Lagrangian under a continuous group of transformations leads to the conservation of the corresponding Noether current. Quantization affiliates a quantum current to this classical Noether current, and it may happen, that unlike the classical current, the quantum current is not conserved but has an anomalous non vanishing divergence. In other words, it may be impossible to maintain a classical conservation law in the corresponding quantum field theory. Examples of this situation are provided by the anomaly of the axial current in massless fermion theories and by the trace anomaly, the anomaly of the dilatation current. If the invariance of the Lagrangian is explicitly broken already at the classical level, the classical and quantum Ward identities will differ by the anomalous contribution.

In this chapter we shall show [7) 19) 27)] that for the large class of currents, including the axial current the anomalous divergence of the quantum current has a topological meaning reflecting topological properties of the underlying classical theory. The Atiyah Singer theorem will be crucial in revealing this topological relevance of the anomaly and will, moreover, provide a powerful tool for calculating the anomalous divergence of a given current.

In the sequel the classical as well as the quantized field theory will always be Euclidean, and the final physical results will be obtained by Wick rotating at the very end.

We consider the following [19)] quite general classical field theory. The Euclidean classical fields are sections of a vector bundle E over a compact Riemannian space time manifold. (In a later section we shall see that the assumed compactness of Euclidean space-time is only a technical assumption.) Denoting the Euclidean field by ψ we furthermore assume that the classical Lagrangean is of the form [19)]

$$L = \langle \psi , D\psi \rangle + \ldots \qquad (4.1)$$

where $\langle .,. \rangle$ is some hermitean scalar product, which we need not specify for the moment and D: $\Gamma(E) \longrightarrow \Gamma(E)$ is an elliptic operator of the first order. The terms omitted in (4.1) may contain additional interactions without derivatives of ψ and contributions from other Euclidean fields. The assumed ellipticity of D just reflects the physical requirement that the kinetic operator is invertible off-shell, D may contain gauge fixing terms to assure this. We could as well have started out with a Lagrangean bilinear in the derivative:

$$L = \langle D\psi , D\psi \rangle \qquad (4.1a)$$

the anomalous divergences which will emerge later on would in fact be identical, but for the definiteness we stay with (4.1).

Next we assume that L be invariant under transformations

$$\psi \longrightarrow e^{\alpha \Gamma_5} \psi \qquad (4.2a)$$

where Γ_5 is a bundle homomorphism (locally a possibly space time dependent matrix) with the additional properties

$$\Gamma_5 D + D\Gamma_5 = 0 \qquad (4.2b)$$

$$\Gamma_5^{\,2} = 1 \qquad (4.2c)$$

Finally, although this is not necessary, we shall for convenience assume that the elliptic operator D is of the form

$$D = \Gamma_i \nabla_i \qquad (4.3)$$

where ∇_i is the covariant derivative in the direction i.

A classical massless spinor field in curved Euclidean space-time possibly coupled to an additional Yang Mills field is an example of the above described situation, and the notations allude to this standard example, but there are other interesting realizations of the general scheme.

The conserved classical Noether current is

$$J_i = \langle \psi, \Gamma_i \Gamma_5 \psi \rangle \quad . \qquad (4.4)$$

Γ_5 is involutive, hence, the fields $\psi \in \Gamma(E)$ can be split into even and odd parts under Γ_5:

$$\Gamma_5 \psi_\pm = \pm \psi_\pm \qquad (4.5a)$$

$$\Gamma(E) = \Gamma_+(E) \oplus \Gamma_-(E) \qquad (4.5b)$$

and (4.2b) implies that D transforms even fields into odd fields and odd fields into even fields. So, by restriction we can define elliptic operators

$$\begin{aligned} D_+ &: \Gamma_+(E) \longrightarrow \Gamma_-(E) \\ D_- &: \Gamma_-(E) \longrightarrow \Gamma_+(E) \end{aligned} \qquad (4.6)$$

which are adjoints of one another.

Now we can state the main assertion of this paragraph [19]:
The divergence of quantum current $\tilde{J}_i$, affiliated to J_i has an anomaly A(x), a polynomial in the curvature quantities of the bundles involved, which is given by minus twice the index density of the elliptic operator D_+. (The index density is the curvature polynomial

on the right hand side of the Atiyah-Singer index theorem (1.7), which has to be integrated over M to give the index.)

In the rest of this section we shall have to justify this vital theorem.

In the quantized theory, the classical current (4.4) is replaced by a functional average (vacuum expectation value of the quantum current operator:

$$\tilde{J}_i = \langle\!\langle J_i \rangle\!\rangle = \int D\psi\, J_i\, e^{-S_E} \Big/ \int D\psi\, e^{-S_E}$$

$$= \frac{\partial}{\partial A_i} \ln \int D\psi\, e^{-S_E + A_i J_i} \Big|_{A=0} \qquad (4.7)$$

$$\overset{\mathrm{Def}}{=} \frac{\partial}{\partial A_i} \Gamma(A) \Big|_{A=0}$$

The integration is a functional integration, the derivative with respect to the external source is a functional derivative, $A_i I_i$ is a short hand notation for $\int_M dx I_i(x) A_i(x)$, and the Euclidean action is

$$S_E = \int_M \mathcal{L}\, dx = \int_M \langle \psi, D\psi \rangle\, dx \qquad (4.8)$$

The vacuum functional $\Gamma(A)$ is formally given by

$$\begin{aligned} \Gamma(A) &= -\ln\det(D - A_i \Gamma_i \Gamma_5) \quad ,(+\ln \det \text{ for fermions}) \\ &= -\ln\det D^{-1} + A_i \operatorname{tr} \Gamma_i \Gamma_5 D^{-1} + \ldots \end{aligned} \qquad (4.9)$$

Hence, in one loop order one has formally

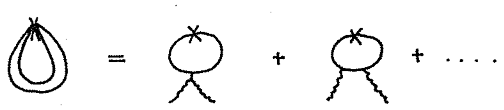
$$\tilde{I}_i = \operatorname{tr} \Gamma_i \Gamma_5 D^{-1} \qquad (4.10)$$

Here, D^{-1} is the Green's function of the operator D.
As it stands, the expression (4.10) is of course ill-defined, and there are various equivalent procedures to regularize and define it.

On can start out from the expression

$$\tilde{I}_i = \lim_{x' \to x} \operatorname{tr}(\Gamma_i(x) \Gamma_5(x) G(x,x'))\Big|_{\text{regularized}} \qquad (4.11)$$

or, graphically



(4.12)

Here the double line represents the propagator of ψ in an external field, the simple line the free ψ propagator, the wavy line the external field and the cross the insertion $\Gamma_i \Gamma_5$.

The task is to calculate $\nabla_i \tilde{I}_i$ in order to find possible anomalies This can be done by [10)]

a) Feynman graph methods [10,28)], where one treats the problem in the momentum space and regularizes the emerging Feynman graphs in a standard way. The anomaly will arise from the lowest graphs depicted in (4.12).

b) Covariant point splitting [19)]. Here one determines the Green's function G for instance, by Schwinger's proper time method and regularizes by dropping terms which are singular in the coincidence limit.

c) ζ - function regularization [27,10,4)].

This method most clearly reveales the relationship between anomalies and the index theorem.

The one-loop vacuum functional is formally given by

$$\Gamma = -\frac{1}{2} \sum_{\lambda} \ln \frac{\lambda^2}{\mu^2} \tag{4.13}$$

where $\{\lambda^2\}$ are the eigenvalues of the non-negative operator $D_A^2 = (D - \Gamma_i \Gamma_5 A_i)^2$, and μ is some regularization mass. Comparing with eq. (2.7) one sees that it is natural to <u>define</u> [4)] Γ by

$$\Gamma = \frac{1}{2} \left[\zeta'(s) + \ln \mu^2 \zeta(s) \right]_{s=0} \quad . \tag{4.14}$$

ζ (s) is the ζ-function of the operator D_A^2. This expression exists for large real part of s, and the value at s = 0 is obtained by analytic combination. For the divergence of $\tilde{I}_i$ one finds (compare 4.11.)

$$\nabla_i \tilde{J}_i(x) = -2\, tr\, \zeta(s,x,x)\, \Gamma_5 \Big|_{A=0} \Big|_{s=0} \qquad \text{(comp. eq. 2.6)}$$

$$= -2 \frac{1}{\Gamma(s)} \int dt\, t^{s-1}\, tr\, \Gamma_5 \left(K(t,x,x) - P_o(x,x) \right) \Big|_{A=0} \Big|_{s=0} \tag{4.15}$$

$$= -2\, tr\, \Gamma_5\, \mu_o(x) + 2\, tr\, \Gamma_5\, P_o(x,x) \quad .$$

$\mu_o(x)$ is the coefficient of t^0 in the asymptotic expansion of the heat kernel of the operator D^2.

Now D^2 commutes with Γ_5, and hence, its heat kernel splits into an even and an odd contribution

$$h = h_+ + h_- \; ; \quad \Gamma_5 h_\pm = \pm h_\pm \tag{4.16}$$

$h_\pm$ are the heat kernels restricted to $\Gamma_\pm(E)$ (compare (4.6)) and belong to the operators

$$\begin{aligned} D_- D_+ &= D_+^* D_+ \qquad \text{and} \\ D_+ D_- &= D_+ D_+^* \end{aligned} \tag{4.17}$$

With this splitting we get

$$\nabla_i \tilde{I}_i(x) = -2\mathrm{tr}\left\{\mu_+(x) - \mu_-(x)\right\} + 2\mathrm{tr}\left\{P_{o+}(x,x) - P_{o-}(x,x)\right\}$$

in evident notation, and looking at (4.17) and (2.13) we see that the anomalous part of the divergence of $\tilde{I}_i$, a curvature polynomial, which is present even if there are no zero modes of D^2, is really given by (-2) times the index density of the operator D_+.

We see, that the anomaly of the current T^i being related to the index of D_+ really has a topological meaning. In contrast to this, for the trace anomaly [4], the anomalous trace of the energy momentum tensor in one loop order the ζ-regularization procedure gives

$$(T_i{}^i)_{\text{Anomalous}} = -2\, \mathrm{tr}\, \mu_o(x) = -2\, \mathrm{tr}\left(\mu_o^+ + \mu_o^-\right) \quad . \tag{4.19}$$

This is again a polynomial in the relevant curvature quantities, but this combination does not in general represent a closed differential form and a characteristic class, but has only a metrical meaning. Related to this, the trace anomaly, unlike the anomaly of the axial current is known to be renormalized in higher loop orders.

It is fair to say that we can resume our findings about the topological significance of the anomalies of quantum currents in the following way:

The anomalous part of the divergence of the quantum current $\tilde{I}_i$ is introduced by the regularization procedure, which includes the extraction of the zero modes. The zero modes may show an asymmetry, (different number of even and odd zero modes) which via the Atiyah-Singer index theorem has topological significance.

Apart from this insight into the nature of current anomalies, which turn out to convey a topological message the gain of what we have derived is twofold:

a) We have a powerful tool to compute anomalies quickly by topological methods.
b) We can use our insight to construct new anomalous currents [19], whose anomalies are related to fundamental topological features of the underlying classical theory.
Both points will be illustrated by examples in the next section.

5. Evaluation of Anomalies

We shall now give several realizations [19] of the general framework described by eqs. (4.1) and (4.2). Throughout we shall assume that the space-time manifold M is of even dimension n, compact, oriented and Riemannian.

We denote the Riemannian metric by g. There exist local n-bein tangent vector fields (e_i) $i=1,\dots,n$, such that $(e_1,\dots,e_n)$ defines the orientation of M and

$$g(e_i,e_j) = \delta_{ij}. \tag{5.1}$$

In the standard way we have a unique torsion free Riemannian connection in the tangent bundle, which we write as

$$\begin{aligned} \nabla_x e_i &= \Omega_{ij}(x)\, e_j \\ \Omega_{ij}(x) + \Omega_{ji}(x) &= 0 \end{aligned} \tag{5.2}$$

with associated Riemannian curvature form R.

From the tangent bundle one can go over to the Clifford bundle C(M) by forming the Clifford algebra of the tangent spaces at all points $m \in M$ with respect to the form $g(m)$.
As local sections of C(M) we shall denote the n-bein fields e_i by γ_i.
Because of the orientability of M

$$\gamma_5 = \gamma_1 \gamma_2 \cdots \gamma_n \tag{5.3}$$

is globally defined.

Our first example is the axial anomaly. Herefore we assume that M has a spin structure (the vanishing of the second Stiefel-Whitney class is a necessaryand sufficient condition for this). Then it is possible to construct a spinor bundle $\Delta(M)$ over M, whose sections are $s = 1/2$ spinor fields, and which splits into an even and an odd part under γ_5:

$$\Delta(M) = \Delta^{+}(M) \oplus \Delta^{-}(M) \tag{5.4}$$

On M we have a canonical spinor connection by

$$\hat{\nabla}_X \psi = X\psi + \frac{1}{4}\Omega_{ij}(X)\gamma_i\gamma_j\psi \tag{5.5}$$

and a globally defined Dirac operator D with

$$D\psi = \gamma_i \hat{\nabla}_{e_i}\psi \quad . \tag{5.6}$$

If for the scalar product in the fibres $\langle .,. \rangle$ we take the ordinary scalar product on spinors we have all the data required for the construction (4.1) and (4.2), because really

$$\begin{aligned} D\gamma_5 + \gamma_5 D &= 0 \\ \gamma_5^2 &= 1 \end{aligned} \tag{5.7}$$

So, in this case E is the spinor bundle $\Delta(M)$, the anomalous current is the axial current

$$J_i = \langle \psi, \gamma_i\gamma_5\psi \rangle \tag{5.8}$$

and the anomaly of the axial current is given by the index density of the operator

$$D_{\pm}: \quad \Gamma\Delta^{+}(M) \longrightarrow \Gamma\Delta^{-}(M) \quad . \tag{5.9}$$

By the methods described in appendix 2, this quantity can be readily determined: it is the $\hat{A}$ genus density [6,25] known from the mathematical literature and given up to terms of degree four by

$$\hat{A} = 1 - \frac{1}{24}p_1 + \dots \quad . \tag{5.10}$$

Here p_1 is the so-called first Pontrjagin class [13], a characteristic class of degree four, which in terms of the Riemannian curvature is given by the closed 4-form [25]

$$p_1 = -\frac{1}{8\pi^2}\mathrm{tr}\, R \wedge R \tag{5.11a}$$

with

$$R_{ij} \quad = \quad \frac{1}{2} \, R_{ijkl} dx^k \wedge dx^l \quad . \tag{5.11b}$$

So, we can immediately evaluate the gravitational part of the axial anomaly [19], which would, for instance lead to a decay of the π°-meson into two gravitons. The ordinary Adler-Bell-Jackiw anomaly [26] is also obtained [7], if one adds a gauge degree of freedom of the Fermion fields, which mathematically corresponds to tensorially multiplying with a vector bundle V_G with structural group G:

$$\Delta^{\pm}(M) \longrightarrow \Delta^{\pm}(M) \otimes V_G \tag{5.12}$$

Using the multiplicativity of ch:

$$ch(E \otimes F) \quad = \quad ch\, E \cdot chF \tag{5.13}$$

and looking at the index theorem (1.7) with
$E = \Delta^{+}(M) \otimes V$ and $F = \Delta^{-}(M) \otimes V$ one gets the index density

$$chV \cdot \hat{A}(M) \quad = \quad -\dim V \, \frac{1}{24} \, p_1 + \frac{\tilde{C}(V)}{4\pi^2} \, F \wedge F \quad . \tag{5.14}$$

In conventional notation the total axial anomaly on a four dimensional Riemannian manifold is given by

$$\nabla_i J_i \quad = - \frac{\dim V}{384\pi^2} \, \varepsilon^{abcd} \, R_{abst} R_{cdst} \quad + \quad \frac{\tilde{C}(V)}{4\pi^2} \, \varepsilon_{ijkl} \, trF_{ij} F_{kl} \tag{5.15}$$

where dim V is the dimension of the representation r(G) under which the spinor field transforms and $\tilde{C}(V)$ is an easily computable number, which only depends on r(G).
For dimension n=4 the two contributions are additive, in higher dimensions also mixed terms will arise.

So we succeed to get the Yang Mills and the gravitational part of the axial anomaly from the index theorem almost without calculation. Especially the evaluation of the gravitational part by conventional methods is quite laborious, and there was a long standing discrepancy about its coefficient in the literature [30].

Next we shall construct two new anomalous currents [19]. For the bundle E over M we take the bundle

$$\Lambda(M) \quad = \quad \bigoplus_{r=0}^{n} \Lambda^{r}(M) \tag{5.16}$$

The sum of all complexified exterior powers of the cotangent bundle of M. The sections of this bundle $\Lambda(M)$ are just all the complex-valued differential forms on M, not necessarily of homogeneous degree. The dimension of $\Lambda(M)$, and hence the number of complex components of the field ψ is 2^n. For instance, for dimension n=4, ψ has 16 components, namely

1 from 0-forms :	scalar
4 from 1-forms :	vector
6 from 2-forms :	antisymmetric tensor of rank two
4 from 3-forms :	pseudovector
1 from 4-forms :	pseudoscalar

Decomposing into spins we see that for n=4, ψ contains spins not greater than one.

On the cotangent bundle of M we have the Hodge star-involution [11,6], because M is oriented, which in terms of the dual forms e^i of the n-dim fields

$$e^i(e_j) = \delta_{ij} \tag{5.17}$$

is given by

$$* e^{i_1} \wedge \ldots \wedge e^{i_r} = \frac{1}{(n-r)!} \varepsilon_{i_1 \ldots i_r j_1 \ldots j_{n-r}} e^{j_1} \wedge \ldots \wedge e^{j_{n-r}} \quad . \tag{5.18}$$

Hence:

$$* : \Lambda^r(M) \longrightarrow \Lambda^{n-r}(M) \tag{5.18a}$$

The scalar product on the fibres of $\Lambda(M)$ is defined in the following way [6]:

$$\langle \alpha, \beta \rangle = \bar{\alpha} \wedge * \beta \tag{5.19}$$

if α and β are homogeneous of equal degree
$\langle \alpha, \beta \rangle = 0$, if α, β are homogeneous of unequal degree and by linear extension if α or β are nonhomogeneous.
There is also a hermitean form

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle \tag{5.20}$$

which turns out to be positive definite.

On $\Lambda(M)$ we have the ordinary exterior derivative

$$d: \Lambda^r(M) \longrightarrow \Lambda^{r+1}(M) \tag{5.20a}$$

with $$d^2 = 0 \tag{5.20b}$$

And by
$$(\alpha, d\beta) = (\delta\alpha, \beta) \tag{5.21a}$$

we can define the adjoint δ of d with respect to the hermitean form

$$\delta: \Lambda^r(M) \longrightarrow \Lambda^{r-1}(M) \tag{5.21b}$$

with $$\delta^2 = 0 \tag{5.21c}$$

which is explicitely given by

$$\delta\alpha = (-1)^{np+n-1} * d * \alpha \quad \text{for } \alpha \in \Gamma\Lambda^p(M) \quad . \tag{5.21d}$$

For the operator D we take the Hodge operator [6) 24)]

$$D = d + \delta \tag{5.22}$$
$$D: \Gamma\Lambda(M) \longrightarrow \Gamma\Lambda(M)$$

which is hermitean and elliptic as can easily be shown.
The kernel of D is spanned by the <u>harmonic</u> differential forms, which are defined by

$$d\omega = \delta\omega = 0 \quad . \tag{5.23}$$

The Hodge decomposition theorem [24)] says that every de Rham cohomology class has precisely one harmonic representative, which implies that the p^{th} Betti number $b_p(M)$ is equal to the number of linearly independent harmonic differential forms of degree p.
Now all other data for the Lagrangean (4.1) are known.
There remains to define the operator Γ_5.
We give two different non-equivalent variants:
a) We define [19)]

$$\Gamma_5^E \omega = (-1)^p \omega \quad , \quad \omega \in \Gamma\Lambda^p(M) \tag{5.24}$$

an operator which is +1 on forms of even degree and -1 on forms of odd degree

b) $\Gamma_5^H \omega = (-1)^{p(p+1)/2 + 1} * \omega$ for $\omega \in \Gamma\Lambda^p(M)$. (5.25)

Here we have assumed that n= dim M = 41 is divisable by four. One verifies

$$(\Gamma_5^{E,H})^2 = 1$$
$$\Gamma_5 D + D\Gamma_5 = 0 \quad . \tag{5.26}$$

Using the identity

$$d\omega = \sum_i e^i \wedge \nabla_{e_i} \omega \tag{5.27}$$

we can write the Hodge operator D in the form [19)]

$$D = (d + \delta) = \Gamma_i \nabla_i \qquad (\nabla_i = \nabla_{e_i}) \tag{5.28a}$$

with

$$\Gamma_i \omega = (e^i \wedge + (-1)^{np+n+1} * e^i \wedge *)\omega \qquad \omega \in \Gamma\Lambda^p(M) \quad . \tag{5.28b}$$

The $2^n \times 2^n$ matrices Γ_i fulfill the same anticommutation rules as the γ_i, and $\{\Gamma_i, \Gamma_5^{E,H}\} = 0$.
The bundle $\Lambda(M)$ splits into even and odd parts under $\Gamma_5^{E,H}$:

$$\Gamma_5^E: \quad \Lambda(M) = \Lambda^e(M) \oplus \Lambda^o(M) \tag{5.29a}$$

$$\Gamma_5^H: \quad \Lambda(M) = \Lambda^+(M) \oplus \Lambda^-(M) \quad . \tag{5.29b}$$

Altogether, the Lagrangean is

$$\mathcal{L} = \langle \omega, \Gamma_i \nabla_i \omega \rangle + \dots \tag{5.30a}$$

and the conserved classical currents are [19)]

$$J_i^{E,H} = \langle \omega, \Gamma_i \Gamma_5^{E,H} \omega \rangle$$

The anomalies of the corresponding quantum currents $\tilde{J}_i^{E,H}$, for which we introduce [19)] the names Euler current and Hirzebruch current, are completely determined by the index densities of the operators

$$D_+^E : \Gamma\Lambda^e(M) \longrightarrow \Gamma\Lambda^o(M)$$
$$D_+^H : \Gamma\Lambda^+(M) \longrightarrow \Gamma\Lambda^-(M) \tag{5.31}$$

Let us first discuss D_+^E. The meaning of ind D_+^E is readily identified [6,24]. ind D_+^E = dim ker D_+^E - dim ker D_+^{E*}.

Now ker D_+^E and ker D_+^{E*} are the spaces of harmonic differential forms of even and odd degree respectively. Thus

$$\dim \ker D_+^E = \sum b_{2r}(M)$$
$$\dim \ker (D_+^E)^* = \sum b_{2r+1}(M) \tag{5.31a}$$

and

$$\text{ind } D_+^E = \sum (-1)^p b_p(M) = \chi(M) \quad , \tag{5.31b}$$

the Euler characteristic of M. $\chi(M)$ has another simple meaning [31]: If M is triangulated with m_i i-simplices, then

$\chi(M) = \sum (-1)^i m_i$, independent of the triangulation.

The right hand side of the Atiyah-Singer index theorem for D_+^E gives the so-called Euler class e, a characteristic class of degree n = dim M, which, when integrated over M yields the Euler characteristic $\chi(M)$. Inverting its well known expression [25] in terms of the Riemannian curvature we finally obtain the anomaly [19] for n=4

$$\nabla_i \tilde{J}_i^E = \frac{1}{64\pi^2} \varepsilon^{ikrs} \varepsilon^{uvab} R_{ikuv} R_{rsab} \quad . \tag{5.32}$$

This result has also been checked by comparing it with the result of a covariant point splitting calculation.

We now come to the operator D_+^H:

ker D_+^H and ker D_+^{H*} are the spaces of harmonic forms even and odd under Γ_5^H respectively. Every harmonic form $\alpha \in \Gamma\Lambda^p(M)$ can be written as

$$\alpha = \frac{1}{2}(\alpha + \Gamma_5^H \alpha) + \frac{1}{2}(\alpha - \Gamma_5^H \alpha) = \alpha_+ + \alpha_- \quad , \tag{5.33}$$

$\alpha_\pm \in \Lambda^\pm(M)$ are harmonic. Hence, for $p \neq 2l$ (remember n = dim M = 4l) the spaces ker $D_+^H \cap \Lambda^p(M)$ and ker $D_+^{H*} \cap \Lambda^p(M)$ are isomorphic, and ind D_+^H gets only contributions from $\Lambda^{2l}(M)$, which is stable under Γ_5^H. On $\Lambda^{2l}(M)$ we have

$$\Gamma_5^H \omega = * \omega \quad . \tag{5.34}$$

In the space of harmonic 2l forms $H^{2l}(M) \subset \Lambda^{2l}(M)$ we introduce a basis $\{\omega_i\}$ such that

$$*\omega_i = \Gamma_5^H \omega_i = \begin{cases} \omega_i & \text{for } 1 \leq i \leq r \\ -\omega_i & \text{for } r+1 \leq i \leq r+s = b_{2l}(M) \end{cases} \tag{5.35}$$

Then

$$\mathrm{ind}\, D_+^H = r - s \qquad (5.36)$$

This is at the same time the signature of the symmetric quadratic form

$$[\alpha,\beta] = \int_M \alpha\wedge\beta = (\alpha,\Gamma_s\beta) \qquad (5.37)$$

on $H^{2l}(M)$, which is by definition [6,24,29] the signature of the manifold M.

For the operator D_+^H the index density is given [6,29] by the L genus, a characteristic class whose parts up to degree four are

$$L = 1 + \frac{1}{3}p_1 + \dots \qquad (5.38)$$

It is the content of Hirzebruch's famous signature theorem [29] that the integral over the L class equals the signature of the manifold, which explains the name of Hirzebruch current for J_i^H.[19] Using (5.11) and (5.38) the anomaly of J_i^H is readily obtained [19]. For example for n=4:

$$\nabla_i J_i^H = \frac{1}{48\pi^2}\,\varepsilon^{ikrs} R_{ikuv} R_{rsuv} \qquad (5.39)$$

Again, this result has also been verified by a covariant point splitting calculation [19].

It would be straightforward to endow the 2^n component field with an additional gauge degree of freedom, which would just like for the spinor field result in multiplying the index density by ch V_G and yield an easily computable Yang -Mills contribution to the anomalies of J_i^E and J_i^H.

We have succeeded in constructing anomalous currents [19], whose anomalies are related to the most fundamental invariants of the space-time manifold M, the Euler characteristic and the signature. For n=4 these currents incorporate only fields of spin $\leqslant 1$. They may play a role in quantum gravity comparable to the importance of the axial current in Yang-Mills theories.

As a last example we shall briefly treat the axial anomaly [28,33,10] of Supergravity [32]. This is a theory, in which a massless spin 3/2 field is coupled to the gravitational field. It is unique in the sense that the usual causality problems, which arise, if a field of higher spin is coupled to the gravitational field are avoided in a consistent way.

The Lagrangean of supergravity is [32)]

$$\mathcal{L} = -\frac{1}{2}\sqrt{g}\,R - \frac{1}{2}\varepsilon^{\lambda\mu\nu\rho}\,\bar{\psi}_\lambda \gamma_\rho \gamma_5 \nabla_\mu \psi_\nu \quad . \tag{5.40}$$

It is invariant under the chiral transformation of the Rarita-Schwinger spinor ψ_μ

$$\psi_\mu \longrightarrow e^{-i\alpha\gamma_5}\,\psi_\mu \tag{5.41a}$$

which leads to a conserved classical Noether current

$$j_\mu^5 = -\frac{i}{2}\,\varepsilon^{\rho\mu\sigma\tau}\,\bar{\psi}_\rho \gamma_\tau \psi_\sigma \quad . \tag{5.41b}$$

The Rarita-Schwinger field ψ_μ still contains redundant unphysical degrees of freedom, which have to be eliminated by subsidiary conditions like

$$\nabla_\mu \psi_\mu = 0 \quad , \quad \gamma_\mu \psi_\mu = 0 \quad . \tag{5.42}$$

This can be done consistently because the Lagrangean eq. (5.40) has a fermionic gauge invariance under the substitution

$$\psi_\mu \longrightarrow \psi_\mu + \nabla_\mu \chi \tag{5.43}$$

and some of the constraints can be interpreted as gauge fixing conditions.

In a field theoretical treatment a gauge fixing term is added to the supergravity Lagrangean, and the unphysical degrees of freedom are compensated by ghost fields. A careful analysis shows [34)], that the ghost fields are spinor fields [33,34)], two with positive and one with negative chiral charge. This result can be guessed by counting degrees of freedom and by looking at the structure of the subsidiary conditions (5.42). The gauge fixing term helps to render the Euclidean kinetic operator elliptic. Its precise form does not matter for the topological evaluation of the anomaly, because the index theorem (1.7) only depends on the bundles involved.

The contribution of the ghost fields has to be subtracted to get the correct axial anomaly.

In geometrical terms, the Rarita-Schwinger field is a section of the bundle

$$\begin{aligned} \mathcal{R} &= \mathcal{R}^+ \oplus \mathcal{R}^- = \Delta(M) \otimes (TM \otimes \mathbb{C}) \\ &= \Delta^+(M) \otimes (TM \otimes \mathbb{C}) \oplus \Delta^-(M) \otimes (TM \otimes \mathbb{C}) \quad , \end{aligned} \tag{5.44}$$

where $TM \otimes \mathbb{C}$ is the complexified tangent bundle of the compact space-time manifold M, corresponding to the additional four-vector index of the Rarita-Schwinger field ψ_μ.

For the calculation of the axial anomaly one needs the Chern character $\mathrm{ch}\, R^+ - \mathrm{ch}\, R^-$. Looking at (5.44) and using the multiplicativity of ch one finds

$$\mathrm{ch}\, R^+ - \mathrm{ch}\, R^- = \mathrm{ch}(TM \otimes \mathbb{C})\left[\mathrm{ch}\,\Delta^+(M) - \mathrm{ch}\,\Delta^-(M)\right] , \tag{5.45a}$$

and from an inspection of the index theorem (1.7) one infers that the index density of the elliptic Rarita-Schwinger operator is obtained from the Dirac density by simply multiplying with

$$\mathrm{ch}(TM \otimes \mathbb{C}) = 4 + p_1 + \cdots . \tag{5.46}$$

(For a derivation of this equality see Appendix II).
The ghost bundle is given by

$$\mathcal{G}^\pm = \Delta^\pm \oplus \Delta^\pm \oplus \Delta^\mp . \tag{5.47}$$

In the index theorem, the contributions of two of the ghost spinor bundles will compensate, and the final axial anomaly will be obtained by simply subtracting once the spin 1/2 index density from the Rarita-Schwinger index density. So, finally one has to multiply the spin 1/2 anomaly density by $(3 + p_i + \ldots)$ to arrive at the desired result. Comparing with the spin 1/2 anomaly we find

$$\begin{aligned} &\mathrm{ind}\, D_e^+ - \mathrm{ind}\, D_{1/2}^+ = (3 + p_1 + \ldots)(1 - \tfrac{1}{24}p_1 + \ldots)[M] = \\ &(3 + \tfrac{21}{24}p_1)[M] = = -21\ \mathrm{ind}\, D_{1/2}^+ . \end{aligned} \tag{5.48}$$

The axial anomaly of the spin 3/2 field is -21 times [33,10] the corresponding spin 1/2 result. Supposing, one can also add a Yang-Mills degree of freedom in a consistent way, also this contribution to the spin 3/2 axial anomaly can easily be evaluated [10]. The results , normalized to spin 1/2 are given in the following table

Spin	Gravitational part	Yang-Mills part
1/2	1	1
3/2	-21	3

More general spinor fields and additional gauge degrees of freedom have been treated in [43].

This evaluation of the supergravity anomaly especially clearly reveals the power of the topological approach to the anomaly problem. Ref [10] contains a more detailed discussion of the spin 3/2 anomaly including a comparison with other methods for computing anomalies.

We remark here that different values for the spin 3/2 axial anomaly can be obtained by evaluating the relevant Feynman graph [28,10] using noncovariant gauge fixing and assuming that the Adler-Rosenberg method of exploiting "gravitational conservation" is applicable in this case. This, however, turns out not to be true [42].

6. Non Compact Spaces

On noncompact spaces or compact spaces with boundaries the index theorem (1.7) cannot be true without modifications.

To illustrate this we give two examples of noncompact Riemannian manifolds, which can be thought to be obtained by letting a boundary tend to infinity.

a) Hawking's Euclidean Taub-NUT space [35] is a Riemannian space with length element

$$ds^2 = \frac{R+m}{R-m}\, dR^2 + 4(R^2 - m^2)\left\{\sigma_x^2 + \sigma_y^2 + \frac{4m^2}{(R+m)^2}\,\sigma_z^2\right\} \quad , \tag{6.1}$$

$$m \leq R < \infty \quad . \tag{6.1a}$$

Here

$$\begin{aligned} \sigma_x &= \tfrac{1}{2}(-\cos\psi\, d\theta - \sin\theta \sin\psi\, d\varphi) \\ \sigma_y &= \tfrac{1}{2}(\sin\psi\, d\theta - \sin\theta\cos\psi\, d\varphi) \\ \sigma_z &= \tfrac{1}{2}(-d\psi - \cos\theta\, d\varphi) \end{aligned} \tag{6.2a}$$

$$0 \leq \varphi \leq 2\pi \quad ; \quad 0 \leq \theta \leq \pi \quad ; \quad 0 \leq \psi \leq 4\pi \tag{6.2b}$$

are the Maurer Cartan forms of the three-sphere S^3 in terms of Euler angles. The singularity of the metric at $R = m$ is only an apparent one and comes only from the polar coordinates. The total topology of the Euclidean Taub-Nut space is simply $\mathbb{R}^4$. Restricting to $m \leq R \leq R_0$ one obtains manifolds with boundary. The boundary surfaces at $R = R_0$ are topological three-spheres with a non-standard metric, which gets more and more asymmetric for $R_0 \to \infty$ because the coefficient of σ_z^2 decreases relative to the coefficient $\sigma_x^2 + \sigma_y^2$.

The Riemannian curvature of the metric (6.1) is self dual, which makes the Euclidean Taub-NUT space a good candidate for a gravitational instanton [35]. To show that the single unmodified index theorem cannot hold for this space we compute the Pontrjagin number

$$p_1[M] = -\frac{1}{8\pi^2}\int_M \operatorname{tr} R\wedge R \tag{6.3}$$

and find

$$p_1[M] = 2 \quad . \tag{6.4}$$

Eq. (5.10) then tells us that using the index theorem (1.7) this would imply for the index of the Dirac operator

$$\text{"ind } D^+_{1/2}\text{"} \quad = \quad -\frac{1}{12} \tag{6.5}$$

an absurd fractional result. Hawking presented this situation as a puzzle [35], and we shall soon exhibit the necessary modifications of the index theorem to render this index an integer.

As a second example we take the

b) Eguchi-Hanson space [36], a Riemannian space with metric

$$ds^2 = dR^2(1 - (\tfrac{a}{R})^4)^{-1} + R^2\left\{\sigma_x^2 + \sigma_y^2 + (1 - \tfrac{a}{R})^4\sigma_z^2\right\} \ . \tag{6.6}$$

The singularity at R=a can be removed by identifying antipodal points. The angle ψ then varies only in the interval $0 \leq \psi \leq 2\pi$. The surfaces at $R = R_o$ are therefore not spheres but real projective three-spaces $\mathbb{RP}(3)$, which arise from S^3 by identifying antipodal points. The metric on these projective spaces tends to the standard metric for $R_o \rightarrow \infty$. Again, for (6.6) the Riemannian tensor is self dual, and this time one finds

$$p_1[M] = -3 \tag{6.7a}$$

and

$$\text{ind } D^+_{1/2} \quad = \quad \frac{1}{8} \ . \tag{6.7b}$$

The metric is asymptotically flat, which makes it a still better candidate for a gravitational instanton. Later on we shall deal with more general boundaries than S^3 and $\mathbb{RP}(3)$, namely with so-called Lens spaces [31], which we are now going to describe.

We start out from the three-sphere S^3, which we realize as the following subset of the complex two dimensional space $\mathbb{C}^2$:

$$S^3 = \left\{ (z_1, z_2) \in \mathbb{C}^2 \,\middle|\, |z_1|^2 + |z_2|^2 = 1 \right\} \quad . \tag{6.8}$$

On S^3 we consider the action of a cyclic group G of order m. The action on $\mathbb{C}^2$ and S^3 is determined by specifying the action of a generator $g_o \in G$:

$$g_o(z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_1) \quad . \tag{6.9}$$

Identifying equivalent points under the action of G we arrive at the Lens space

$$L(\theta_1, \theta_2) \overset{\text{Def}}{=} S^3/G \tag{6.10}$$

in particular

$$\mathbb{RP}(3) = L(\pi, \pi) \quad . \tag{6.11}$$

We now state an index theorem of Atiyah, Singer and Patodi [20] for compact Riemannian manifolds with boundary. Our noncompact manifolds are obtained by pushing the boundary to infinity. For definiteness we state it for the Dirac operator. One has [20]

$$\text{ind } D^+_{1/2} = \hat{A}[M] + K - \frac{1}{2}(\eta_{1/2}[\partial M] + h_{1/2}) \quad . \tag{6.12}$$

The first term $\hat{A}[M]$ is the same integral over a curvature polynomial which appears on the right hand side of the index theorem for compact manifolds without boundary. This by the way justifies that we restricted ourselves to compact manifolds without boundary in section 4 and 5. The second term K is a local boundary term, an integral over the boundary of a differential form, which depends on the curvature tensor and the second fundamental form. It is the so-called transgression [25] form of the class $\hat{A}$. K vanishes if the manifold is a Riemannian product and in the two examples tends to zero, if R_o of the boundaries $R = R_o$ goes to infinity. We need not further discuss this contribution.

The last contribution

$$\xi_{1/2} = \frac{1}{2} (\eta_{1/2} [\partial M] + h_{1/2}) \tag{6.13}$$

is of particular interest. It can be shown to depend only on the boundary ∂M and not on the interior $\mathring{M}$.

It persists also for $R_0 \to \infty$ and will provide the missing fractional part of the index. $\xi_{1/2}$ is a very peculiar spectral invariant of the boundary, defined in the following way:

Introducing normal and tangential coordinates in the neighbourhood of the boundary ∂M on can write the Dirac operator $D^+ : \Delta^+_{1/2}(M) \longrightarrow \Delta^-_{1/2}(M)$ as

$$D^+ = \gamma_n \nabla_n + \gamma_\alpha \nabla_\alpha = \gamma_n (\nabla_n + A) \quad . \tag{6.14}$$

The operator A is hermitean and can be identified with the Dirac operator on the boundary ∂M.

Let

$$h_{1/2} = \dim \ker A, \tag{6.15}$$

be the number of zero modes of the operator A. Denoting the eigenvalues of A by $\{\lambda\}$ one defines

$$\eta_{1/2}(s) = \sum_{\lambda \neq 0} \operatorname{sign} \lambda \, |\lambda|^{-s} \tag{6.16}$$

and

$$\eta_{1/2}[\partial M] = \eta(0) \tag{6.17}$$

in the sense of an analytic continuation.

This strange quantity, obtained from the spectrum of the Dirac operator A on ∂M vanishes if the eigenvalues λ are distributed symmetrically about $\lambda = 0$. It describes an asymmetry of the eigenvalue distribution.

For the Euclidean Taub-NUT space the boundary contribution $\xi_{1/2}$ can be evaluated directly [21,38] diagonalizing the Dirac operator on S^3 with nonstandard left-invariant metric.

The result for the boundary contribution of the boundary at $R_0 \to \infty$ is[21,22]

$$\text{Taub-NUT} \quad : \quad \xi_{1/2} = -1/12 \tag{6.18a}$$

and hence

$$\operatorname{ind} D^+_{1/2} = -\frac{1}{12} + \frac{1}{12} = 0 \quad . \tag{6.18b}$$

This vanishing of the index means that no chirality is lost by tunneling of a $s = 1/2$ spinor Taub-NUT background field.

The evaluation of the boundary contribution for the Eguchi-Hanson space is performed [23] with the aid of the G-index theorem. Rather than evaluating $\xi_{1/2}$ on $S^3/G = L(\Theta_1, \Theta_2)$ with standard metric, the boundary of D^4/G, where $D^4 = \{x \in \mathbb{R}^4 \mid |x| \leq 1\}$, we can undo the identification modulo G and consider the G-index on D^4 with boundary S^3 instead. The origin $0 \in \mathbb{R}^4$ is the only fix point of the action of G, and the boundary correction for this case is given by the right hand side of the G index theorem (1.16) and the(vanishing) g-index for the Dirac operator on D^4 with flat metric.

This, in turn, yields the boundary correction of the Lens spaces: [23,39]

$$\xi_{\frac{1}{2}}(\Theta_1, \Theta_2) = \frac{1}{4m} \sum_{k=1}^{m-1} \frac{1}{\sin\frac{k\Theta_1}{2} \cdot \sin\frac{k\Theta_2}{2}} \quad . \qquad (6.19)$$

The above sketched procedure is described in detail for the signature operator instead of the Dirac operator in ref. [20]. For more details see ref. [41].

In particular, for the Eguchi-Hanson space we obtain with $\Theta_1 = \Theta_2 = \pi$

$$\text{Eguchi-Hanson:} \quad \xi_{1/2} = 1/8 \qquad (6.20a)$$

and

$$\text{ind } D^+_{1/2} = \frac{1}{8} - \frac{1}{8} = 0 \quad , \qquad (6.21b)$$

again no chirality loss of a spin 1/2 field in the gravitational instanton field.

This absence of vacuum tunneling does not pertain to the spin 3/2 case [23] as was first conjectured by Hawking and Pope [40], who were able to construct two zero modes of the Rarita-Schwinger operator in a background Eguchi-Hanson metric, both of negative chirality, and none of positive chirality.

The G-index theorem provides a rigorous way to evaluate the index for spin 3/2.

The index of the operator $D^+_R : R^+ \longrightarrow R^-$ (compare section 5) is given [10] by a formula analogous to eq. (6.12):

$$\text{ind } D^+_R = \hat{A}_R[M] - \xi_R[\partial M] \quad , \qquad (6.22)$$

where we have already omitted the boundary contribution K, which vanishes for $R_0 \longrightarrow \infty$.

In section 5 we had already obtained [10]

$$\hat{A}_R[M] = \{ \text{ch}(TM \otimes \mathbb{C}) \hat{A} \}[M] = \qquad (6.23)$$

$$= \frac{5}{6}\, p_1[M] \;=\; -\,20\,\hat{A}[M] \tag{6.23}$$

$$= 5/2 \qquad \text{for the metric (6.6)} \quad .$$

The boundary correction ξ_R is in an analogous way obtained [23] from the $s = 1/2$ result by multiplying the G-index density with

$$i^* ch_g\,(TM \otimes \mathbb{C}) = 2\cos k\theta_1 + 2\cos k\theta_2 \quad . \tag{6.24}$$

Here i^* means restriction to the fixed point set $M^g = \{0\}$. The result is for Lens spaces [23]

$$\xi_R = \frac{1}{2m} \sum_{k=1}^{m-1} \frac{\cos k\theta_1 + \cos k\theta_2}{\sin\frac{k\theta_1}{2}\,\sin\frac{k\theta_2}{2}} \tag{6.25}$$

$$= -\frac{1}{2} \qquad \text{for the Eguchi-Hanson space} \quad .$$

Hence

$$\text{ind } D_R^+ \;=\; -\,\frac{5}{2} + \frac{1}{2} \;=\; -2 \tag{6.26}$$

and, taking into account the ghost contributions for the Rarita-Schwinger field (compare 5.48)

$$\text{ind } D_R^+ - \text{ind } D_{1/2}^+ \;=\; -2 - 0 \;=\; -2 \tag{6.27}$$

in agreement with Hawking and Pope [40].

So, for spin 3/2 chirality does get lost by tunnelling. Boundary contributions for higher spinor fields are evaluated in Ref. [43]. The Atiyah-Singer-Patodi index theorem (6.12) is the appropriate framework [21] to treat broken winding numbers, which appear on non compact manifolds without running into inconsistencies for the chirality bilance. Such fractional configurations may be important in a topological theory of quark confinement.

A more detailed description of the G-index evaluation of boundary terms and a discussion of more general gravitational instanton background fields can be found in Ref. [41].

Appendix I: De Rham Cohomology

In this appendix we shall collect a few fundamental definitions and properties concerning the de Rham cohomology of a manifold, the only kind of cohomology modules which is used in this work.

Let M ba a (smooth) manifold. The linear space of all (smooth) differential p-forms on M is denoted by $\Omega^p(M) = \Gamma \Lambda^p(M)$. Furthermore we write

$$\Omega(M) = \bigoplus_{p=0}^{\dim M} \Omega^p(M) \quad . \tag{I, 1}$$

On the differential forms we have an exterior derivative

$$d : \Omega^p(M) \longrightarrow \Omega^{p+1}(M) \tag{I, 2}$$

and an associative exterior multiplication

$$\wedge : \Omega^p(M) \times \Omega^q(M) \longrightarrow \Omega^{p+q}(M) \tag{I, 3}$$

with the well-known properties

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \tag{I,4a}$$

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta \tag{I,4b}$$

$$d^2 \alpha = 0 \tag{I,4c}$$

$$\alpha \in \Omega^p(M) \, , \; \beta \in \Omega^q(M) \quad .$$

In addition, d is of course linear and $\wedge$ bilinear.

A differential form ω is called closed if $d\omega = 0$ and exact, if there exists a form ψ such that $\omega = d\psi$. The spaces of closed and exact p-forms on M are denoted by $Z^p(M)$ and $B^p(M)$. One has, because of (I,4c)

$$B^p(M) \subset Z^p(M) \quad . \tag{I,5}$$

The p^{th} de Rham cohomology group of M is defined as the quotient vector space

$$H^p(M) = Z^p(M)/B^p(M) \quad , \tag{I,6}$$

its elements are equivalence classes of closed p-forms, two closed p-forms being equivalent if their difference is an exact p-form. A smooth map f: $M \longrightarrow N$ induces linear maps

$$f^*: \Omega^p(N) \longrightarrow \Omega^p(M) \qquad (I,7)$$

such that

$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta \qquad (I,8a)$$
$$f^* d\alpha = d f^*\alpha \quad . \qquad (I,8b)$$

In terms of coordinates f^* just means composition with the Jacobian matrix of f. Because of (I,8), the map f also induces a linear map between the cohomologies, which we shall also denote by f^*:

$$f^*: \quad H^p(N) \longrightarrow H^p(M) \qquad (I,9)$$

and one can show that homotopy of two maps f and g implies identity of the associated maps f^* and g^* on the cohomology groups.

We conclude with a few important constructions related to connections on bundles. We shall restrict ourselves to vector bundles.

Let E be a vector bundle over N with standard fibre V and structural group G. Consider a map f: $M \longrightarrow N$. Then one can construct the so-called <u>induced bundle</u> f^*E, a vector bundle over M with the same standard fibre V. We again use the notation f^*, because there is no danger of confusion with the previously introduced mappings f^*. The induced bundle f^*E over M is obtained by taking the fibre of E over f(m) as the fibre over m for f^*E. For details see e.g. ref.[11-13]. Homotopic maps induce equivalent bundles.

Consider now a connection on E with (locally defined) connection form A and curvature $F = dA + 1/2 A \wedge A$. A and F are $\mathfrak{g}$-valued one-and two-forms on N respectively. $\mathfrak{g}$ denotes the Lie algebra of G. Then there is an induced connection on the induced bundle f^*E defined by

$$A^{ind} = f^*A \qquad (I,10a)$$

which implies

$$F^{ind} = dA^{ind} + 1/2 A^{ind} \wedge A^{ind} = f^*F \quad . \qquad (I,10b)$$

A symmetric r-linear form

$$\begin{aligned} Q: \quad & \mathfrak{g} \times \mathfrak{g} \times \ldots \times \mathfrak{g} \longrightarrow \mathbb{C} \\ & (a_1, a_2, \ldots, a_r) \longrightarrow Q(a_1, \ldots, a_r) \end{aligned} \qquad (I,11)$$

is called <u>invariant</u>, if

$$Q([b,a],a,\dots,a) = 0 \quad \text{for all } a,b \in \mathfrak{g} \quad . \tag{I,12}$$

Inserting the curvature form F into a symmetric invariant r-linear form one obtains a 2r-form

$$Q_F = Q(F,F,\dots,F) \quad . \tag{I,13}$$

The precise meaning of (I,14) can be described as follows.
Writing F as $F = F_{\mu\nu}(x)\,dx^{\mu}\wedge dx^{\nu}$ with $F_{\mu\nu}(x) \in \mathfrak{g}$
we have

$$Q_F = Q(F_{\mu_1\nu_1},\dots,F_{\mu_r\nu_r})\,dx^{\mu_1}\wedge dx^{\nu_1}\wedge\dots\wedge dx^{\mu_r}\wedge dx^{\nu_r} \quad . \tag{I,13a}$$

We assert that Q_F is closed and, moreover, the cohomology class of Q_F is independent of the specific connection on E and, hence, only depends on E.

For the first assertion we observe (confer def.(3.2), we drop the subscript of D)

$$dQ_F = DQ_F = rQ(DF,F,\dots,F) = 0 \tag{I,14}$$

because of the Bianchi identity DF = 0 .

To prove the second assertion we have to show that the change δQ_F, due to an infinitesimal change δA of A is an exact form.
Indeed

$$\begin{aligned} \delta Q_F &= rQ(\delta F,F,\dots,F) \\ &= rQ(D\delta A,F,\dots,F) \quad \text{(eq. 3.6)} \\ &= rdQ(\delta A,F,\dots,F) \quad \text{(invariance of Q)} \ . \end{aligned} \tag{I,15}$$

So, the cohomology class $[Q_F]$ of Q_F is really independent of the connection. This implies that for the induced bundle f^*E we have

$$[Q_{f^*F}] = f^*[Q_F] \tag{I,16}$$

and that $[Q_{f^*F}]$ only depends on f^*E.

Appendix II Characteristic Classes

In this section we confine ourselves to smooth real or complex vector bundles over smooth manifolds. The cohomology will always be de Rham's cohomology which is all we need. For more general cases see e.g. ref. [13].

Consider a vector bundle $E = (\tilde{E}, \pi_E, M_E)$ with total space $\tilde{E}$, base M_E and projection π_E. A characteristic class χ is a function which associates to E a cohomology class of the basis

$$\chi : \; E \longrightarrow \chi(E) \in H^*(M_E) \tag{II,1a}$$

with

$$H^*(M_E) = \bigoplus_p H^p(M_E) \quad . \tag{II,1b}$$

In addition one requires that χ behaves naturally with respect to induced bundles

$$\chi(f^*E) = f^*\chi(E) \tag{II,1c}$$

Eqs. (II,1) define the notion of a characteristic class.

For instance, the considerations at the end of appendix I and formula (I,16) tell us that for vector bundles with a fixed structure group G every invariant n-linear symmetric form Q defines a characteristic class χ_Q by

$$\chi_Q(E) = Q(F,\ldots,F), \tag{II,2}$$

where F is the curvature form of any connection on the bundle E.

Equation (II,1c) shows that trivial bundles have trivial values for all characteristic classes, because a trivial bundle can always be induced from a bundles whose base is a point.

The most fundamental characteristic classes of <u>complex</u> vector bundles are the Chern classes $c_i(E)$, which are uniquely defined by (II,1) and the following properties:

$$c_i(E) \in H^{2i}(M_E) \tag{II,3a}$$
$$c_o(E) = 1 \quad , \quad c_i(E) = 0 \text{ for } i > \dim E$$

(Here dim E is the dimension of the fibres of E. We define

$$c(E) = \sum_{i\geq 0} c_i(E))$$
$$c(E\oplus F) = c(E)c(F) \tag{II,3b}$$

The multiplication on the right hand side is an exterior multiplication, commutative, because the $c_i(E)$ are forms of even degree. (II,3b) is equivalent with

$$c_i(E\oplus F) = \sum_{k\geq 0} c_k(E)\, c_{i-k}(F) \quad . \tag{II,3c}$$

A normalization is fixed by prescribing the Chern classes of the canonical one dimensional vector bundle E(n) over the n-dimensional complex projective spaces $\mathbb{C}P(n)$. The fibre over a point of $\mathbb{C}P(n)$ is just the one dimensional complex vector space which represents this point. One defines

$$c(E^{(n)}) = 1 + x \tag{II,3d}$$

where $x \in H^2(\mathbb{C}P(n))$ is the positive integer generator of the de Rham cohomology of $\mathbb{C}P(n)$. Axioms (II,3) in particular guarantee that the integrals of the Chern classes over closed compact surfaces are always integer valued.

Assume now that the complex bundle E is the sum of complex line bundles (one dimensional bundles):

$$E = \bigoplus_{j=1}^{\dim E} E_j \tag{II,4}$$

and call $c_1(E_j) = y_j$. Then (II,3b) implies

$$c(E) = \prod (1 + y_j) \tag{II,5}$$

and $c_k(E)$ is just the k^{th} elementary symmetric polynomial of the quantities y_j. Not every bundle E is splittable into line bundles, but one can prove the splitting principle [6,29], which states that from every complex vector bundle E one can induce a bundle f^*E such that f^*E is splittable and the map $f^*: H^*(M_E) \longrightarrow H^*(M_{f^*E})$ is injective. Hence it suffices to consider splittable bundles. Formally every complex bundle can be treated as a splittable bundle. In terms of such formal splittings one can conveniently define the other characteristic classes which are used in this work. For instance

Todd classes

$$td\ E = \sum td_i E = \prod \frac{y_i}{1-e^{-y_j}} = 1 + 1/2C_1(E) + 1/12(C_2(E) + C_1^2(E)) + \ldots \qquad (II,6)$$

where $td_i E$ is the homogeneous part of degree i on the right hand side.

Chern character

$$ch\ E = \sum ch_i E = \prod e^{y_j} \qquad (II,7)$$

one verifies at once

$$td\ (E \oplus F) = tdE\ tdF \qquad (II,8a)$$

$$ch(E \oplus F) = chE + ch\ F \qquad (II,8b)$$

$$ch(E \otimes F) = chE \cdot ch\ F \quad . \qquad (II,8c)$$

The last equation (II,8c) holds because for line bundles L_k one has

$$c_1(L_i \otimes L_k) = c_1(L_i) + c_1(L_k) \quad . \qquad (II,8d)$$

Of particular importance is the case that the complex bundle E arises from complexification of a real bundle:

$$E = E' \otimes \mathbb{C} \qquad (II,9)$$

where E' is a real bundle. If E is of even dimension we can always assume [6,29] that the splitting $E = \oplus E_i$ can be chosen such that the classes y_j are pairwise equal up to a sign:

$$y_{2j-1} = -y_{2j} = x_j \qquad (j = 1 \ldots 1/2\dim E) \quad . \qquad (II,10)$$

On can furthermore achieve that in terms of the k_j classes x_j the Euler class, which we define for real orientable even dimensional bundles is given by

$$e(E') = \prod x_j \quad . \qquad (II,11)$$

The Pontrjagin classes are defined for real vector bundles E' by

$$p_i(E') = (-1)^i c_{2i}(E' \otimes \mathbb{C}), \text{ hence} \qquad (II,12b)$$

$$\sum p_i(E') = \prod{}' (1 + x_i^2) \quad . \tag{II,12b}$$

For convenience we list some of the characteristic classes, which appear in this work in terms of the formal splitting of the complexified tangent bundle $TM \otimes \mathbb{C}$ of an 2r dimensional manifold:

$$\mathrm{ch}\ (TM \otimes \mathbb{C}) = \sum_{i=1}^{r} (e^{x}\ re^{-x}) = 2r + p_1(TM) + \dots \tag{II,13a}$$

$$e\ (TM) = \prod_{i=1}^{r} x_i \tag{II,13b}$$

$$\mathrm{td}\ (TM \otimes \mathbb{C}) = \prod_{i=1}^{r} \frac{(-x_i^2)}{(1-e^{-x_i})(1-e^{x_i})} \tag{II,13c}$$

$$\mathrm{ch}\,\Lambda^{+}M - \mathrm{ch}\,\Lambda^{-}M = \prod_{i=1}^{r} (e^{-x_i} - e^{+x_i}) \tag{II,13d}$$

$$\mathrm{ch}\,\Delta^{+}M - \mathrm{ch}\,\Delta^{-}M = \prod_{i=1}^{r} (e^{x_i/2} - e^{-x_i/2}) \tag{II,13e}$$

$$L(M) = \prod_{i=1}^{r} \frac{x_i}{\tanh x_i} = 1 + \frac{1}{3} p_1(TM) + \dots \tag{II,13f}$$

$$\hat{A}(M) = \prod_{i=1}^{r} \frac{x_i/2}{\sinh x_i/2} = 1 - \frac{1}{24} p_1(TM) + \dots \tag{II,13g}$$

We now define the equivariant Chern character ch_g for the complex vector bundles. We only need it for bundles over fixed point sets of the group element $g \in G$, i.e. for bundles on the base of which g acts trivially and only permutes inside the fibres. Again we can apply a formal splitting [6] $E = \oplus E_j$ and assume furthermore that g acts on E_i simply by multiplication with $e^{i\theta_j}$. Then one has

$$\mathrm{ch}_g\ E = \sum e^{i\theta_j + y_j} \tag{II,14}$$

with $y_j = c_1(E_j)$.

In particular, in terms of the formal splitting of the complexified (co) tangent bundle and for zero dimensional fixed point sets we have

$$i^{*}\mathrm{ch}\ g\ (\Lambda_{-1}N^{g} \otimes \mathbb{C}) = i^{*} \prod_{j=1}^{r} (1 - e^{x_j + i\theta_j})(1 - e^{-x_j - i\theta_j}) \tag{II,15a}$$

$$= \prod_{j=1}^{r} (1 - e^{i\theta_j})(1 - e^{-i\theta_j})$$

$$i^* ch\, g\, (TM\otimes\mathbb{C}) = i^* \sum_{j=1}^{r} (e^{x_j + i\theta_j} + e^{-x_j - i\theta_j}) \qquad (II,15b)$$

$$= \sum_{j=1}^{r} 2\cos\theta_j$$

$$i^* chg(\Delta^+(M) - \Delta^-(M)) = i^* \prod_{j=1}^{r} (e^{x_j/2 + i\theta_j/2} - e^{-x_j/2 - i\theta_j/2}) \qquad (II,15c)$$

$$= \prod_{j=1}^{r} (e^{i\theta_j/2} - e^{-i\theta_j/2})$$

i^* here means restriction to the fixed point set M^g.

Finally we note that the Chern classes and, hence all the characteristic classes (II,13), which are polynomials in the Chern classes can be represented by curvature quantities in the form χ_Q of eq.(II,2). For an s x s matrix X we define the invariant polynomials $Q_j(x)$ by

$$\det(\mathbb{1} + \frac{i}{2\pi} X) = \sum_j Q_j(X)\, t^{s-j} \quad . \qquad (II,16)$$

For a complex vector bundle with fibre dimension s one introduces a connection, whose curvature form F is an s x s matrix of two-forms (on every vector bundle there are infinitely many connections). Then one can prove [25)]

$$c_j(E) = Q_j(F) \qquad (II,17)$$

independent of F. This formula allows the expression of characteristic classes by curvature quantities, which was used so frequently in our work. The only case not yet covered by eq. (II,17) is the Euler class. For the Euler class of the tangent bundle of an even dimensional orientable Riemannian manifold we have [25)]

$$e(TM) = \frac{(-1)^{n/2}}{2^n \pi^{n/2} (\frac{n}{2})!} \varepsilon_{i_1 \ldots i_n} R_{i_1 i_2} \wedge \ldots \wedge R_{i_{n-1} i_n} \qquad (II,18)$$

where R_{ik} is the Riemannian curvature tensor of M.

References

1) M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld, Y.I. Manin, Phys. Lett. 65A , 185 (1978)

2) I.M. Singer, Proceedings of the International Congress of

Mathematicians, Vancouver 1974.
M. Kac, Amer. Math. Monthly 73, 1-23 (1966)

3) H.B. G. Casimir, Proc. Kon. Ned. Akad. Wetenschap 51, 793 (1948), see also T.H. Boyer, Ann. Phys. (N.Y.) 56, 474 (1979) and ref. 5)

4) S.W. Hawking, Comm. Math. Phys. 55, 133 (1977)

5) B.S. De Witt, Physics Reports 19C, 295 (1975)
S.M. Christensen, Phys. Rev. D14, 2490 (1976); D17, 976 (1978)

6) M.F. Atiyah, I.M. Singer, Bull. Amer. Math. Soc. 69, 422 (1969)
Amer. Math. 87, 484 and 546 (1968)
M.F. Atiyah, G.B. Segal, Ann. Math. 87, 530 (1968)
R. Palais ed. "Seminar on the Atiyah-Singer Index Theorem", Annals of Math. Studies 57, Princeton University Press.

7) S. Coleman, unpublished
J. Kiskis, Phys. Rev. D15, 2329 (1977)
R. Jackiw, C. Rebbi, Phys. Rev. D16, 1052 (1977)
N.K. Nielsen, B. Schroer, Nuclear Phys. B 127, 493 (1977)
N.K. Nielsen, H. Römer, B. Schroer, Phys. Lett. 70B, 445 (1977)

8) M.F. Atiyah, R. Bott, V.K. Patodi, Inventiones math. 19, 279 (1973)

9) N.K. Nielsen, Nordita preprint 78/24

10) M.T. Grisarn, N.K. Nielsen, H. Römer, P. van Niewenhuizen, Nucl. Phys. B140 477 (1978),
N.K. Nielsen and H. Römer, Proc. Informal Meeting on recent developments in field theory Nov 1977 ICTP/77/152, pp. 27-28.

11) e.g. R.L. Bishop, R.J. Crittenden, "Geometry of Manifolds" Academic Press, New York and London 1964

12) N. Steenrod, " The Topology of Fibre Bundles", Princeton University Press.
D. Husemoller, "Fibre Bundles", Springer New York, Heidelberg, Berlin.

13) J.W. Milnor, J.D. Stasheff, "Characteristic Classes", Annals of Math. Studies Nr.76 Princeton University Press

14) A.S. Schwarz, Phys. Lett. 67B, 172 (1977)

15) R. Jackiw, C. Rebbi, Phys. Lett. 67B, 189 (1977)

16) M. Atiyah, N. Hitchin, I.M. Singer, Proc. Nat. Acad. Sci. USA, 74, 2662 (1977); Proc. R. Soc. Lond. A362, 425 (1978)

17) C.W. Bernard, N.H. Christ, A.H. Gerth, E. Wienberg, Phys. Rev. D16, 2967 (1977)

18) H. Römer, Nuevo Cim. Lett 21. 381 (1978)

19) N.K. Nielsen, H. Römer, B. Schroer, Nucl. Phys. B136, 475 (1978)

20) M.F. Atiyah, V.K. Patodi, I.M. Singer, Bull. Londons Math. Soc 5, 229 (1973), Math. Proc. Cambridge Philos. Soc. 77, 43 (1975);

78, 405 (1975); 79, 71 (1976)
21) H. Römer, B. Schroer, Phys. Lett. 71B, 182 (1977)
22) T. Eguchi, P.B. Gilkey, A.J. Hanson, Phys. Rev. D17, 423 (1978)
C.N. Pope, Nucl. Phys. B141, 432 (1978)
23) A.J. Hanson, H. Römer, Phys. Lett. 80B, 58 (1978)
See e.g.
24) P.B. Gilkey, "Elliptic Operators on Compact Manifolds",
Publish or Perish Inc. Boston 1974 and references herin.
25) See e.g. S. Chern, J. Simons, Proc. Nat. Acad. Sci. (USA) 68, 791 (1971); Ann. of Math. (2) 99, 48 (1974) and Milnor, Stasheff, ref. 13)
26) S.L. Adler, Phys. Rev. 177, 2426 (1969)
J.S. Bell, R. Jackiw, Nuovo Cim. 60A, 47 (1969)
27) J.S. Dowker, J. of Phys. A11, 34 (1978)
28) M.T. Grisarn, P. van Nieuwenhuizen, D.Z. Friedemann, Phys. Lett. 71B, 377 (1978)
29) F. Hirzebruch, "Topological Methods in Algebraic Geometry", Grundlagen der Math. Wissenschaften in Einzeldarstellungen Nr. 131, Springer N.Y. 1966.
30) R. Delbowgo, A. Salam, Phys. Lett. 30B, 381 (1972)
T. Eguchi, P.G.O. Freund, Phys. Rev. Lett. 37, 1251 (1976)
31) See e.g. M. Greenberg, "Lectures on Algebraic Topology", W.A. Benjamin, Inc., Amsterdam, New York 1967
32) S. Deser, B. Zumino, Phys. Lett. 62B, 335 (1976)
For a review see e.g. S. Ferrara, Erice Lectures Aug. 1978) CERN TH 2514, an almost complete list of references is contained in "Proceedings of the Supergravity Workshop at Stony Brook, 27.- 29. Sept. 1979, P. van Nieuwenhuizen, D.Z. Friedman eds.
33) S.M. Christensen, M.J. Duff, Phys. Lett 76B, 571 (1978)
34) N.K. Nielsen, Nucl. Phys. B140, 499 (1978); B142, 306 (1978)
35) S.W. Hawking, Phys. Lett. 60A, 81 (1977)
36) T. Eguchi, A.J. Hanson, Phys. Lett. 74B, 249 (1978)
37) V.A. Belinskii, G. W. Gibbons, D.N. Page, C.N. Pope, Phys. Lett. 76B, 433 (1978)
38) N. Hitchin, Advances in Math. 14, 1 (1974)
39) M.F. Atiyah, unpublished
40) S.W. Hawking, C. N. Pope, Nucl. Phys. B146, 381 (1979)
41) G.W. Gibbons, C. N. Pope, H. Römer, Nucl. Phys. B157, 377 (1979)
42) N.K. Nielsen, Nucl. Phys. B151, 536 (1979)
43) H. Römer, Phys. Lett. 83B, 172 (1979)